Model to explain the Fig. 2 and Fig. 4 data

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Assumptions (widths are always given as FWHM):

- the intrinsic width of the resonance is very small ($w_i \approx 0.059$ kHz) and has a lorentzian (Breit-Wigner) line shape.
- the observed width is mainly given by the spin tune spread, caused by the beam momentum spread. This distribution is assumed to be gaussian. From the data at fixed frequency a width of $w_b \approx 1.6 \pm 0.5$ kHz was deduced, but the analysis of Dieter Prasuhn showed that for a beam momentum spread of $\delta p/p = 2.5e^{-4}$ (standard deviation) on expects $w_b \approx 2.9$ kHz (FWHM).

Thus the observed line shape for the data taken at fixed frequency (Vassilis mail from July 23rd) should be gaussian and given by the convolution integral of the intrinsic line shape (solid black line in Fig. 1)

$$I(f_{\rm rf}) \propto \frac{\frac{w_i^2}{4}}{\frac{w_i^2}{4} + (f_{\rm rf} - f_r)^2} \tag{1}$$

and the spread induced by the beam's momentum distribution (dashed black line in Fig. 1)

$$B(f_{\rm rf} - f_r) \propto \exp\left\{ -\frac{(f_{\rm rf} - f_r)^2}{2w_b^2/(2.35)^2} \right\} \tag{2}$$

so that the observed line shape is given by

$$A(f_{\rm rf}) \propto I * B(f_{\rm rf}) = \int_{-\infty}^{\infty} \mathrm{d}f I(f) \cdot B(f_{\rm rf} - f) \tag{3}$$

(red line in Figs. 1). Apparently the observed line shape is still almost gaussian and has width w_b.

If we now sweep the frequency over some frequency range $f - \Delta f/2 \ldots f + \Delta f/2$ we affect only part of the beam (i.e. only that part of the beam with a momentum matching the spin tune) .

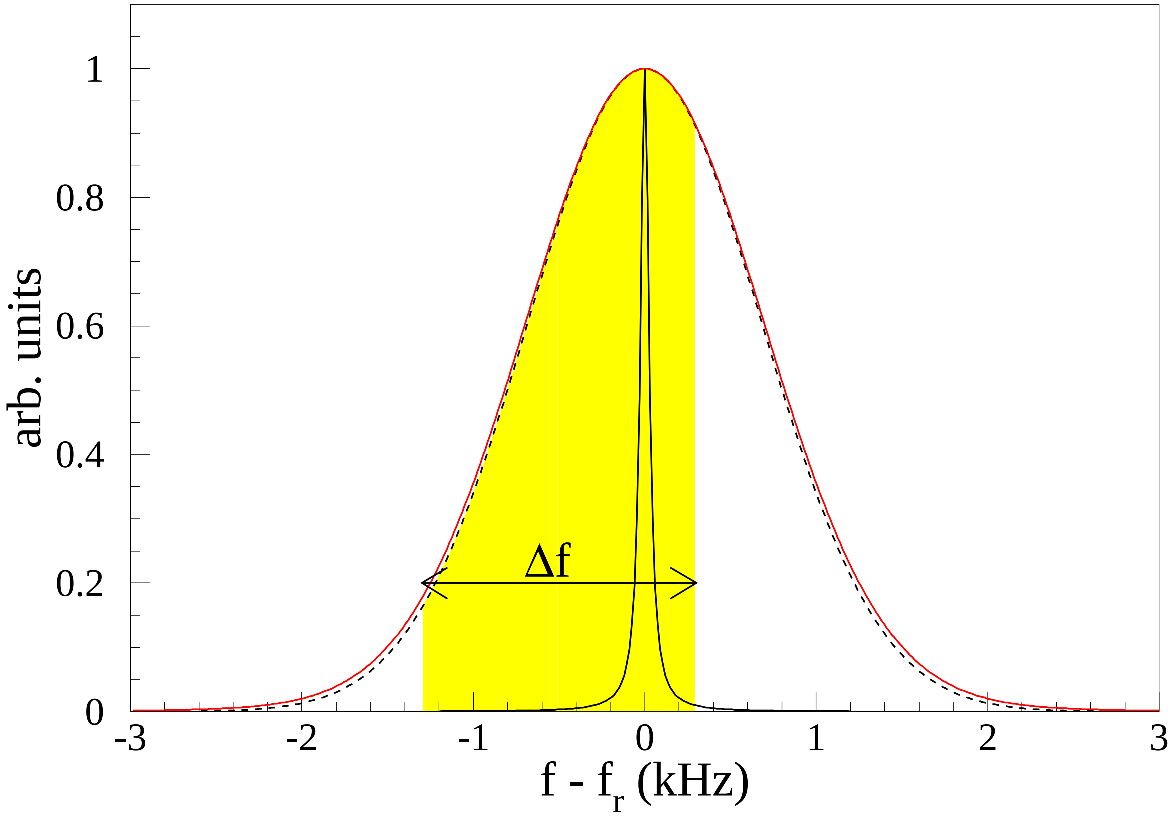

Figure 1: Comparison of different line shapes for $w_b = 2.9$ kHz.

If we neglect the intrinsic width completely and assume that the crossing speed is small enough to completely flip the spin of all particles with a spin tune within the frequency range the probability of a proton to be flipped is the yellow integral in Fig. 1 divided by the total area of the gaussian.

The model views the beam as a collection of beams with a very small momentum spread (smaller than w_i) and their own momentum-dependent resonance frequency. The range and central frequency of the sweep decide which beams flip and which do not. The flip for each beam is described by the Froissart-Stora equation. Note that this is an approximation since the beams with a resonance frequency close to the edges of the frequency range will not flip completely.

Assume for a moment that the central frequency of the sweep is right at the resonance, i.e. Δf is centered symmetric on the gaussian. Note that for a gaussian the probability to be within k standard deviations from the mean is

$$p(|x - \bar{x}| \leq k\sigma) = \mathrm{erf}(k/\sqrt{2}) \tag{4}$$

which describes the spin flip probability and thus

$$p(|x - \bar{x}| > k\sigma) = 1 - \mathrm{erf}(k/\sqrt{2}) \tag{5}$$

is the probability that the particle is unaffected. Here erf(x) is the error-function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{d}t \; e^{-t^2}. \tag{6}$$

k can be related to the width w_b by the following equation.

$$\frac{\Delta f}{2} = k\sigma = \frac{k w_b}{2.35} \quad \Rightarrow \quad k = \frac{2.35 \Delta f}{2 w_b} \tag{7}$$

For Δfnot symmetric around the resonance frequency this generalizes to

$$p_{\mathrm{flip}}(f, \Delta f) = \frac{1}{2} \left\{ \mathrm{erf}\left(\frac{2.35(\mathrm{f_{max}} - \mathrm{f_r})}{\sqrt{2}\mathrm{w_b}} \right) + \mathrm{erf}\left(\frac{2.35(\mathrm{f_r} - \mathrm{f_{min}})}{\sqrt{2}\mathrm{w_b}} \right) \right\} \tag{8}$$

where

$$f_{max} = f + \frac{\Delta f}{2}, f_{min} = f - \frac{\Delta f}{2} \tag{9}$$

This specializes for $f = f_r$ to

$$p_{\mathrm{flip}}(f, \Delta f) = \mathrm{erf}\left(\frac{2.35\Delta\mathrm{f}}{2\sqrt{2}\mathrm{w_b}} \right) \tag{10}$$

With the probability for no spin flip

$$p_{\mathrm{noflip}}(f, \Delta f) = 1 - p_{\mathrm{flip}}(f, \Delta f) \tag{11}$$

we can write the expected final polarization P_f as a function of the initial polarization P_i to be

$$P_f(f, \Delta f) = P_i \cdot p_{\mathrm{noflip}}(f, \Delta f) - P_i \cdot p_{\mathrm{flip}}(f, \Delta f) = P_i[1 - 2 \cdot p_{\mathrm{flip}}(f, \Delta f)] \tag{12}$$

Fig. 2 data

Fig. 2 shows an application of this model to the data of Fig. 2 of the letter. We obtain a perfect representation with only three hand-adjusted parameters: $P_i = 0.8$, $f_r = 1.3060$ MHz and $w_b = 2.7$ kHz. Note that the depth of the dip for Δf=$\pm$1kHz and $\pm$2kHz comes out quite naturally and both data sets are fitted with exactly the same parameters!

A fit of these parameters to the data (with errors adjusted to yield a reduced χ^2 of 1) is shown in Fig. 3. The results

Δf	P_i	f_r (kHz)	w_b (kHz)
$\pm 1 kHz$	0.784 ± 0.298	1306.0 ± 0.6	2.77 ± 0.05
$\pm 2 kHz$	0.801 ± 0.015	1306.0 ± 0.1	2.77 ± 0.10

are consistent for the two frequency ramps.

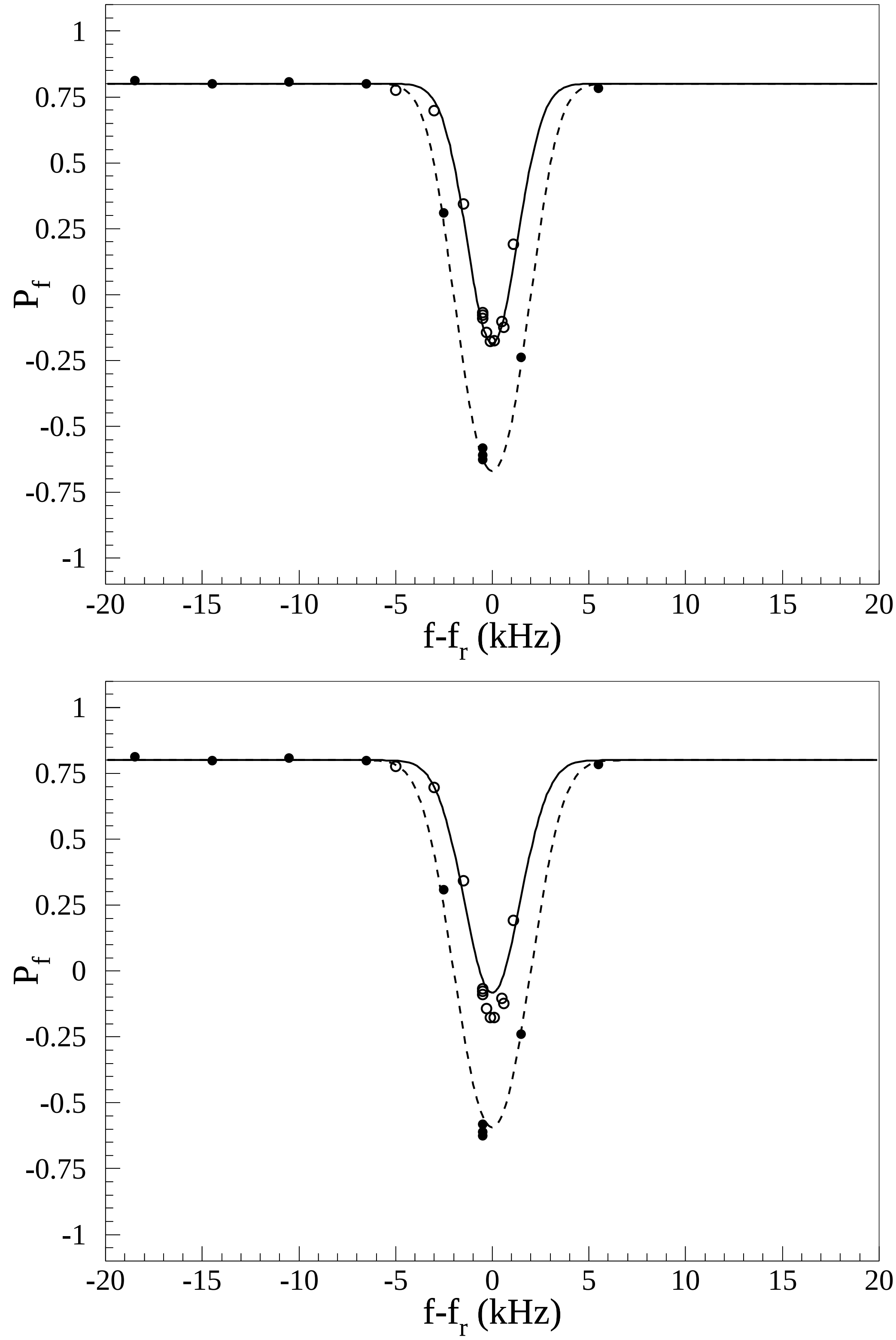

Figure 2: Comparison of the model to Fig. 2 data of the letter using $f_r = 1.3060$ MHz and $P_i = 0.8$. Top: $w_b = 2.7$ kHz (best fit). Bottom: $w_b = 3.1$ kHz (best fit to Fig. 4 data of the letter).

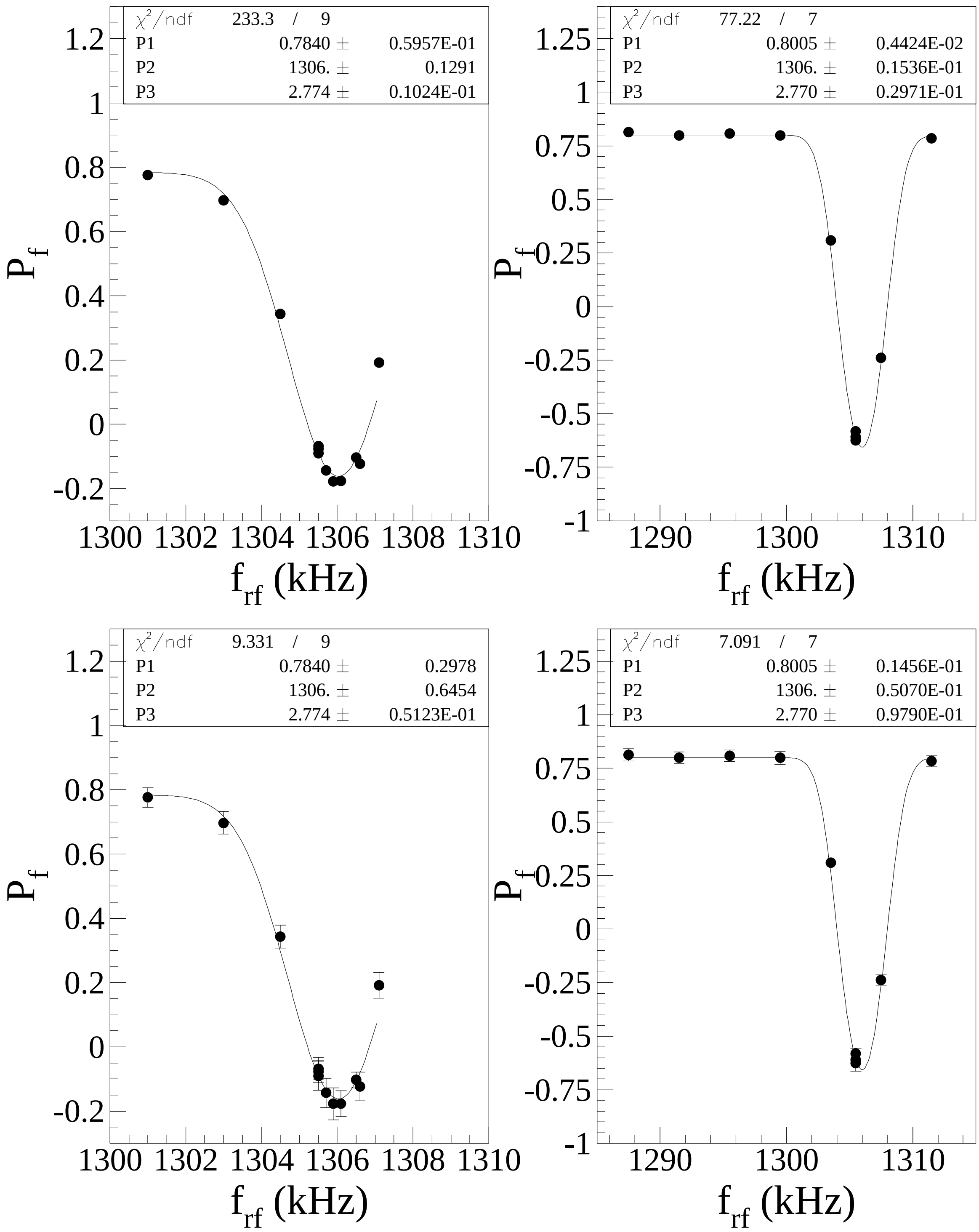

Figure 3: Fit of the model parameters to Fig. 2 data of the letter using using the original errors of the polarization (top) and blown up by factors of 5 (3.3) to yield a reduced χ^2 of 1 for the $\Delta f/2 = 1(2)$ kHz data shown on the left (right).

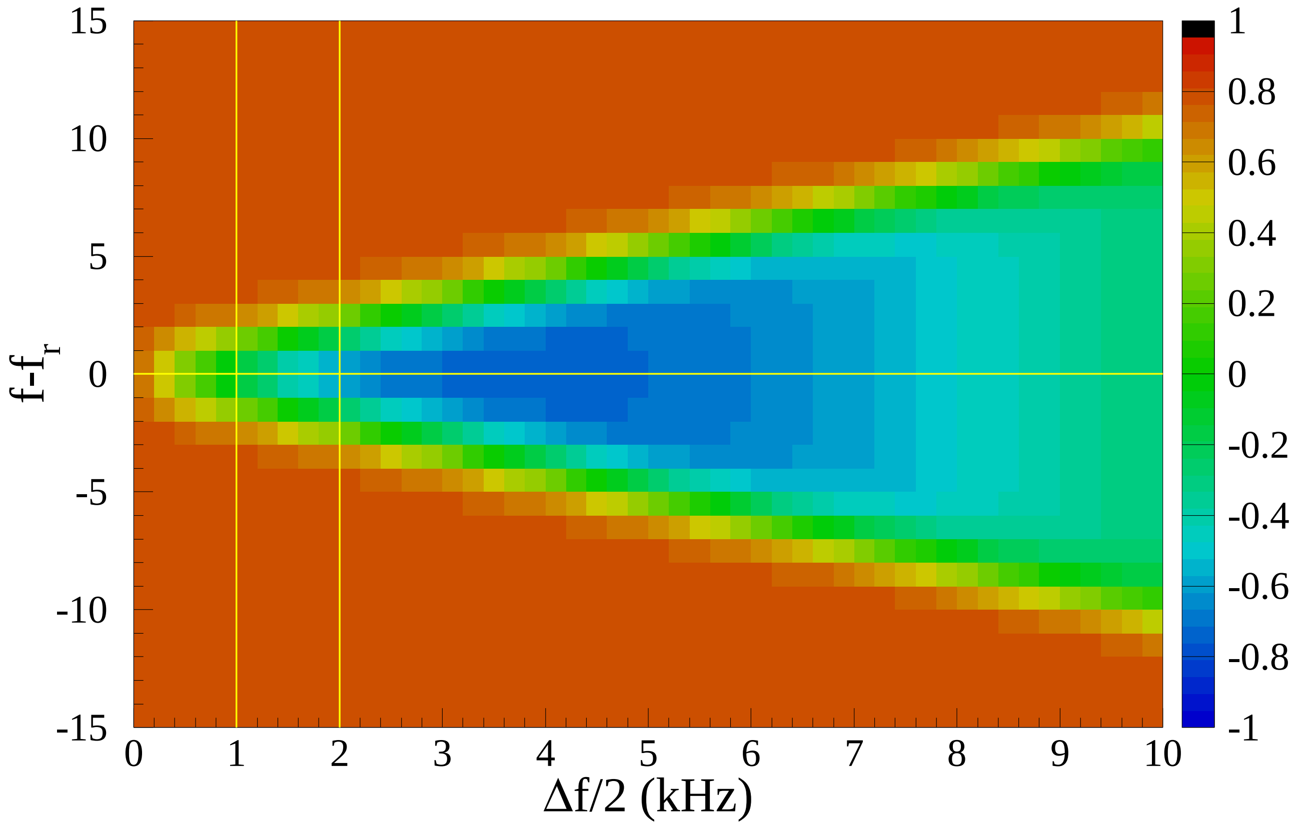

Figure 4: Contour $P_f(\Delta f, f - f_r)$, the lines indicate the measurements of Fig. 2 and 4 of the letter.

Fig. 4 data

Neglecting now the intrinsic width of the resonance, which is small, and assuming that all affected particles make a complete spin flip the final polarization is

$$P_f = P_i \cdot \left\{1 - \text{erf}\left(\frac{k}{\sqrt{2}}\right)\right\} - P_i \cdot \text{erf}\left(\frac{k}{\sqrt{2}}\right) = P_i\left\{1 - 2 \cdot \text{erf}\left(\frac{2.35\Delta f}{2\sqrt{2}w_b}\right)\right\} \tag{13}$$

(assuming the frequency range is centered properly on the resonance frequency)

For large frequency sweeps the spin flip, even of the particles affected by the sweep is not perfect and given by the (modified) Froissart-Stora formula. (One has to replace $-P_i$ in Eq. (12) by P_f of the modified Froissart-Stora formula (Eq. (6) or (8) of the letter). The general expression for $P_f(f, \Delta f)$ can be easily obtained by combining Eqs. (8), (11) and (12), its contour is plotted in Fig. 4)

For n sweeps and $f = f_r$ we obtain:

$$P_f = P_i\left\{1 - \left[1 - \left([1+\eta]\exp\left[\frac{-(\pi\epsilon f_c)^2}{\Delta f/\Delta t}\right] - \eta\right)^n\right] \cdot \text{erf}\left(\frac{2.35\Delta f}{2\sqrt{2}w_b}\right)\right\} \tag{14}$$

The results of Eqs. (13) and (14) are compared to the Fig. 4 data of the letter in Fig. 5

With a value of $w_b = 3.1$ kHz the data of Fig. 4 is perfectly described. However, it is unfortunate that the value for w_b needed disagrees with those derived from the measurements at fixed frequency and that of Fig. 2. ($w_b = 2.7$ kHz). This may be due to the fact that beam parameters have changed slightly between these measurements, e.g. a change in the momentum spread of the beam or the beam momentum and thus the mean resonance frequency (red dotted curve in Fig. 3)

Conclusion

- All data can be explained by the Froissart-Stora equation, when one takes into account, that only part of the beam is affected by the rf-dipole. This happens when the frequency-range is smaller or comparable to the effective resonance width $w_b \approx 2.9 \pm 0.2$ kHz, induced by the beam momentum spread. This number is more or less consistent with the measured beam momentum spread.

- The intrinsic resonance width is much smaller and cannot be accessed experimentally.

- For the beam momentum distribution a gaussian seems more appropriate than a lorentzian, thus, Fig. 2 data should rather be fitted by a gaussian.

- Small corrections will arise for those parts of the beam with a resonance frequency close to the limits of the frequency sweep. Here, a modification of the Froissart-Stora equation needs to be worked out for cases where the (intrinsic) resonance width is not completely covered. The size of this corrections will probably be not more than 10% on the final polarization.

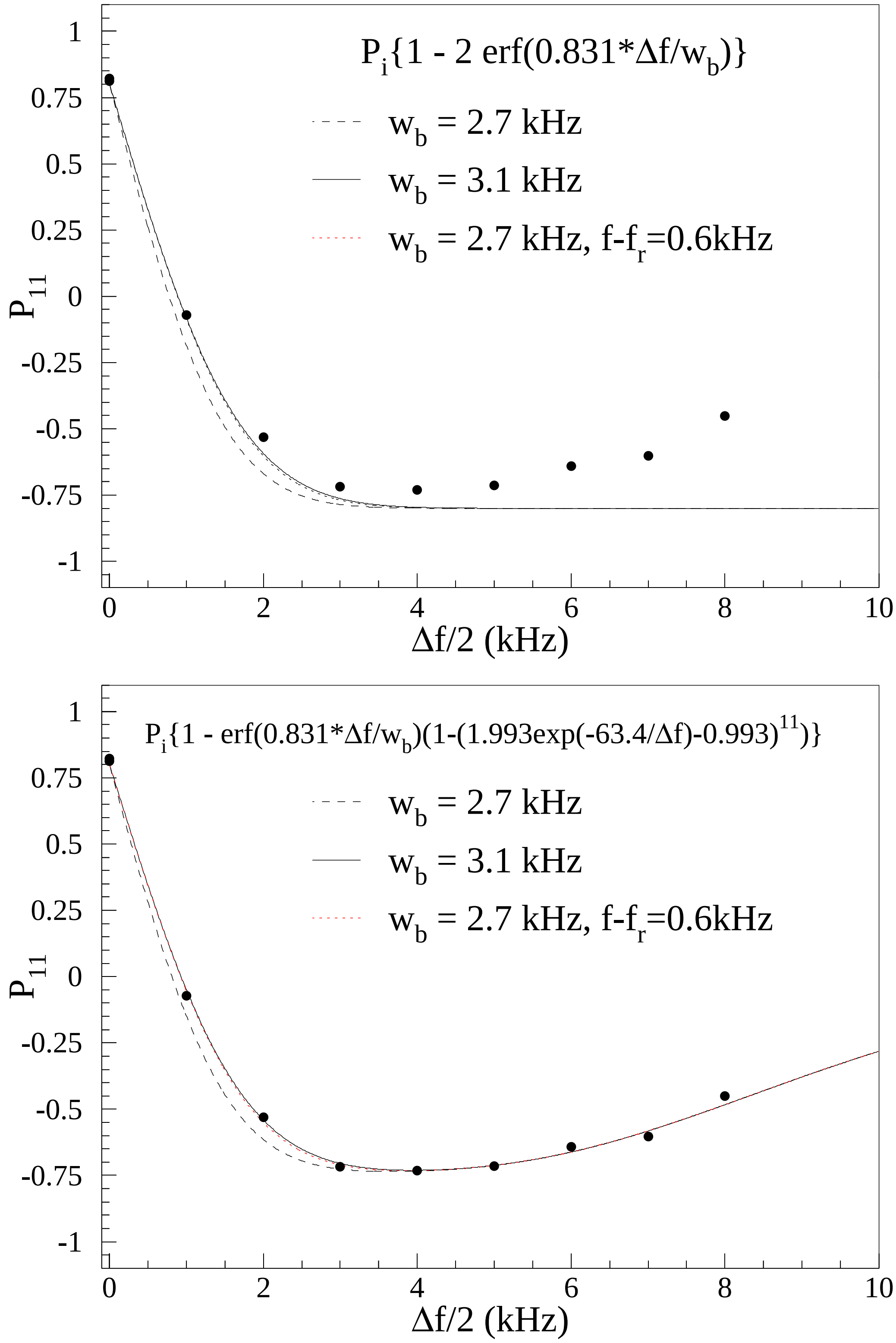

Figure 5: Comparison of the data of Fig. 4 of the letter with the simple model. Top: taking only the spin tune spread into account. Bottom: using the modified Froissart-Stora formula to describe the part of the beam affected by the frequency sweep.